CS131 Homework #4 (18 pts)

Prove the following statements. To prove them, start by writing givens and goals, modifying givens and goals, as discussed in the lectures, and then write the proof reasoning going from givens to goals.

1. Suppose a, b, c, and d are real numbers, 0 < a < b, and d > 0. Prove that if ac ≥ bd then c > d.

|  |  |
| --- | --- |
| Givens | Goals |
| a, b, c, and d are real numbers  0 < a < b  d > 0 | ac ≥ bd → c > d |
| Givens | Goals |
| a, b, c, and d are real numbers  0 < a < b  d > 0  ac ≥ bd | c > d |

Proof: multiplying 2nd given by d we get: ad < bd. Combining with 4th given we get: ad < ac. Since 0 < a we can divide both parts by a and get the goal.

1. Suppose A ⊆ C, and B and C are disjoint. Prove that if x A then x B.

|  |  |
| --- | --- |
| Givens | Goals |
| A ⊆ C  B and C are disjoint  x A | x B |

Proof: x A⊆ C, since B and C are disjoint, we get x B.

1. Suppose that y + x = 2y − x, and x and y are not both zero. Prove that y ≠ 0.

|  |  |
| --- | --- |
| Givens | Goals |
| y + x = 2y – x  ¬ ((x= 0) ˄ (y=0)) | y ≠ 0 |

Proof: assume that y=0. Substituting to the first given, we get x=-x, so x=0. But that contradicts to the 2nd given. Therefore our assumption was incorrect, so y ≠ 0.

1. For all real numbers x and y there is a real number z such that x + z = y − z.

|  |  |
| --- | --- |
| Givens | Goals |
| x and y are any real numbers | ∃z (x + z = y – z) |

Proof. From the goal we get the expression for z: z=(y-x)/2. Substituting it into x+z=y-z we check that z =(y-x)/2 satisfies the goal. Thus, for any x,y, we have found z such that x+z=y-z.

1. For all integers a and b there is an integer c such that a | c and b | c.

(x|y means x divides y)

|  |  |
| --- | --- |
| Givens | Goals |
| a and b - integers | ∃c - integer ( a | c and b | c) |

Proof. Take c = ab. Then c satisfies the goal.

1. If A is non-emptyset and A ⊆ B \ C then A and C are disjoint and B⊈C .

|  |  |
| --- | --- |
| Givens | Goals |
| A is non-emptyset  A ⊆ B \ C | A and C are disjoint  B⊈ C |

Proof.

Let’s prove the first goal. Assume that A and C are not disjoint, so there exists x ∊A⋂C. From the 2nd given x ∊A, implies x ∊B and x∉C. We get a contradiction with our assumption: x ∊A and x ∊C. Therefore A and C are disjoint.

Let’s prove the 2nd goal: assume that B⊆ C. From the 2nd given A ⊆ B, so combining with the assumption we get A⊆ C. But that contradicts to the 2nd given: A ⊆ B and A ⊈ C. Thus. Our assumption was incorrect and B⊈ C.

1. For every integer x, the remainder when x4 is divided by 8 is either 0 or 1.

|  |  |
| --- | --- |
| Givens | Goals |
| x- integer | x4 % 8 =0 or x4 % 8 = 1 |

Proof.

Case 1: x is even (x=2k, k is integer). x4 =16k4 , so x4 % 8 =0

Case 2: x is even (x=2k+1, k is integer). x4 =(2k+1)4 =(4k2+4k+1)(4k2+4k+1)=16(…)+8(…)+1 so x4 % 8 =0

1. For every real number a>0, either a≤1 or f(x)=ax increases with x.

(Hint: function increases if f’(x)>0, (ax)’ = ax ln a )

|  |  |
| --- | --- |
| Givens | Goals |
| real number a>0 | a≤1 or f(x)=ax increases with x |

Proof.

Case 1: If a≤1 does not contradict the givens, and the goal is true.

Case 2: If a>1 then ln a>0, so (ax )’= ax ln a>0, so f(x)=ax increases with x.

1. For every real number x, if | x − 3 | > 3 then x2 > 6x.

(Hint: If x − 3 ≥ 0 then | x − 3 | = x − 3, and if x − 3 < 0 then | x − 3 | = 3 − x. Assume that x − 3 ≥ 0 in case 1, and x − 3 < 0 in case 2; break your proof into cases.)

|  |  |
| --- | --- |
| Givens | Goals |
| | x − 3 | > 3 | x2 > 6x |

Proof.

Case 1: x − 3 ≥ 0. From givens x – 3>3, so x>6. Since x≥ 3, we may multiply both parts by x to get x2 > 6x.

Case 2: x − 3 < 0. From givens 3-x>3, so x<0. x2 >0, 6x<0, so x2 > 6x in this case as well.